High-order differential approximants

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Abstract

We introduce a new form of differential approximant for the summation of power series. The method is a special type of Padé-Hermite approximant. It consists of a high order linear differential equation with polynomial coefficients that is satisfied approximately by the partial sum of the power series. This method is able to reproduce the polylogarithmic functions exactly. Numerical evidence suggests that this is currently one of the best methods of singularity analysis for many problems.

Key words: Differential approximants, Series summation.

1 Introduction

The modelling of physical phenomena often results in nonlinear problems for some unknown function, say $u(\lambda)$. Usually the problems can not be solved exactly. The solutions of these nonlinear systems are dominated by their singularities: physically, a real singularity controls the local behaviour of a solution.

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It is often possible to expand a solution in powers of some parameter. If it is observed that the solution is analytic at some point λ_0, and if we take $\lambda_0 = 0$ for simplicity, then one can solve the problem by expanding the solution in a power series

$$U(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \qquad \text{as} \qquad \lambda \to 0. \tag{1}$$

For physical (nonlinear) problems it is not always possible to find an unlimited number of terms of the power series (1). Often it is possible to obtain a finite number of terms (say N)

$$U_N(\lambda) = \sum_{n=0}^{N-1} a_n \lambda^n \tag{2}$$

of that series (1), and these may contain a remarkable amount of information. One can reveal the solution behaviour near the critical points by analysing the partial sum (2).

The summation of power series is widely used to approximate solutions in many areas of mathematics. Particularly in the study of critical phenomena [8], a very valuable and widely used tool has been the calculation and analysis of series. Over the last quarter century, highly specialized techniques have been used to extract the series coefficients, and at the same time a variety of methods [2] have been devised for extracting the required information of the singularities from a finite number of series coefficients. The most frequently used methods are the ratio-like methods, such as the Domb-Sykes plot [8], Neville-Aitken extrapolation [8], the Hunter & Guerrieri method [11] and seminumerical approximant methods, such as Padé approximants [2], algebraic and differential approximants [2] etc.

We are concerned in this paper with determining the dominant behaviour of the solution by using the partial sum (2). We expect that the accuracy of the critical parameters will ensure the accuracy of the solution. The dominant behaviour of a solution of a linear ODE can often be written as [8]

$$u(\lambda) \sim \begin{cases} C\Big(\lambda_c - \lambda\Big)^{\alpha} & \text{for} \quad \alpha \neq 0, 1, 2, \dots, \\ C\Big(\lambda_c - \lambda\Big)^{\alpha} \ln\Big|\lambda_c - \lambda\Big| & \text{for} \quad \alpha = 0, 1, 2, \dots, \end{cases} \tag{3}$$

as $\lambda \to \lambda_c$, where C is some constant and λ_c is the critical point with the critical exponent α.

To this end we introduce a new differential approximant method. The incentive for designing new approximants is provided by series that cannot be summed accurately by existing methods. The series for the Kelvin–Helmholtz instability is a good example [12,13]. If the new method of approximation can reproduce the dominant singularity of the series, then one can expect the rapid convergence of the approximants.

The structure of this paper is as follows: In § 2, we describe the Padé-Hermite approximants and the differential approximants, in order to clarify our new method. We then give a precise description of our new method, which we call the *high-order differential approximant*. The novel feature is that the order of the differential equation which defines the approximant increases unboundedly with the number of series coefficients used. The main justification of the method is that the approximant is *exact* for many interesting series. Several examples are given in § 3. An application to a problem of fluid dynamics is discussed in § 4. Finally, we conclude with some remarks in § 5.

2 The basic procedure

We start by considering the general form of approximant method known as the Padé-Hermite class. Almost all the one-variable approximants in seminumerical approximant methods belong to this class. In its most general form, this class is concerned with the simultaneous approximation of several independent series. In this paper, we are interested in summing a single series $U(\lambda)$, and we apply the Padé-Hermite approach by taking, the first $d-1$ derivatives of $U(\lambda)$.

Let $d \in \mathbb{N}$ and let the $d+1$ series

$$U_0(\lambda),\, U_1(\lambda),\, \cdots,\, U_d(\lambda)$$

be given. We say that the $(d+1)$-tuple of polynomials

$$\left[A_N^{(0)}, A_N^{(1)}, \cdots, A_N^{(d)}\right],$$

where

$$\deg A_N^{(0)} + \deg A_N^{(1)} + \ldots + \deg A_N^{(d)} + d = N, \tag{4}$$

is a *Padé–Hermite form* for these series if

$$\sum_{k=0}^{d} A_N^{(k)}(\lambda) U_k(\lambda) = O(\lambda^N) \qquad \text{as} \qquad \lambda \to 0\,. \tag{5}$$

For the moment, we concentrate on the practical problem of finding polynomials $A_N^{(k)}$ that satisfy equations (4) and (5). Polynomials are completely determined by their coefficients, so the total number of unknowns in equation (5) is

$$\sum_{k=0}^{d} \deg A_N^{(k)} + d + 1\,.$$

If we now expand the left-hand side of (5) in powers of λ, we see that equation (5) is equivalent to equating the first N terms in the expansion to zero. This gives a system of N equations for the unknown coefficients of the Padé–Hermite polynomials. We obtain an additional equation by imposing some sort of normalisation. For instance, we may require that

$$A_N^{(k)}(0) = 1 \quad \text{for some } 0 \le k \le d\,. \tag{6}$$

The condition (4) then ensures that the number of unknowns exactly matches the number of equations. We may also view it as an optimality condition, in the sense that the sum of the degrees of the polynomials $A_N^{(k)}$ satisfying (5) should be as small as possible. One way to construct the Padé–Hermite polynomials is then to solve the linear system of equations by standard methods such as Gaussian elimination. The computational complexity of this approach is

$$O(N^3) \ \text{(work)}, \quad O(N^2) \ \text{(storage)} \quad \text{as} \quad N \to \infty.$$

Questions of uniqueness and convergence of the Padé–Hermite polynomials in some special cases are treated in [2].

It is important to emphasise that the only inputs required for the calculation of the Padé–Hermite polynomials are the first N coefficients of the series U_0, ..., U_d.

Differential approximants constitute an important member of the Padé–Hermite class. Differential approximant is obtained by taking

$$d \geq 2, \quad U_0 = 1, \quad U_1 = U, \quad U_2 = DU, \ \ldots \quad U_d = D^{d-1}U \tag{7}$$

where D is the differential operator

$$D := \frac{d}{d\lambda}\,.$$

Once the Padé–Hermite polynomials have been found, a *differential approximant u_N* of the series U can then be defined as the solution of the differential equation

$$A_N^{(0)} + A_N^{(1)} u_N + A_N^{(2)} D u_N + \ldots + A_N^{(d)} D^{d-1} u_N = 0\,. \tag{8}$$

In this paper, unless otherwise stated, the name "differential approximant" will refer to the non-homogeneous form of the method, i.e. equations (7) and (8).

Equation (8) is a linear differential equation of order $d-1$ with polynomial coefficients. There are $d-1$ linearly independent solutions, but only one of them has the same first few Taylor coefficients as the given series U. When $d > 2$, the usual method for solving such an equation is to construct a series solution, so it is not immediately clear that anything has been gained by calculating the polynomials $A_N^{(k)}$. Recursive numerical methods can be use to solve Equation (8).

Differential approximants are used chiefly for series analysis. They are powerful tools for locating the singularities of a series and for identifying their nature [9]. In this respect, the key is to note that it is not necessary to *solve* the differential equation (8) in order to find the singularities of u_N. In practice, one usually finds that its dominant singularities are located at zeroes of the leading polynomial $A_N^{(d)}$. Hence, some of the zeroes of $A_N^{(d)}$ may provide approximations of the singularities of the series U. Other features of the singularities can be deduced directly from the polynomials $A_N^{(k)}$.

A less general form of the method of differential approximants was developed by Guttmann & Joyce [9] and Hunter & Baker [11] for series analysis. However, these studies considered only low order differential approximants, where d is not related to N.

When the function has a countable infinity of branches, then the fixed low order differential approximants may not be useful. As noted by Sergeyev & Goodson [17] for algebraic approximants, this suggests that it might be a good idea to let d increase with N, rather than keeping it fixed as $N \to \infty$. Sergeyev & Goodson presented some analysis that identified the choice

$$d = O(\sqrt{N}) \quad \text{as } N \to \infty, \tag{9}$$

as an 'optimal' strategy.

Independently of Sergeyev & Goodson, Drazin & Tourigny [4] had already implemented this idea for algebraic approximants. In other words, N is made to tend to infinity by increasing d.

2.1 A new approach

We introduce here a new member of the Padé–Hermite class in which, as was the case for the Drazin–Tourigny method, the number d of series increases with N. It leads to a particular kind of differential approximant u_N, satisfying Equation (8), where

$$N = \frac{1}{2} d\,(d+3) \tag{10}$$

and

$$\deg A_N^{(k)} = k\,. \tag{11}$$

From (11), we deduce that there are

$$\sum_{k=0}^{d}(k+1) = \sum_{k=1}^{d+1} k = \frac{1}{2}(d+1)(d+2)$$

unknown parameters in the definition of the Padé–Hermite form. In order to determine those parameters, we use the N equations that follow from (5), i.e.

$$A_N^{(0)}(\lambda) + \sum_{k=1}^{d} A_N^{(k)}(\lambda)\, D^{k-1}U(\lambda) = O(\lambda^N) \quad \text{as } \lambda \to 0\,.$$

In addition, we normalise by setting

$$A_N^{(0)}(0) = 1\,. \tag{12}$$

The choice (12) then ensures that there are as many equations as unknowns.

As for other differential approximants, one of the roots, say $\lambda_{c,N}$, of the coefficient of the highest derivative

$$A_N^{(d)}(\lambda_{c,N}) = 0,$$

gives an approximation of the dominant singularity λ_c of the series U. If we assume a singularity of algebraic type as in Equation (3), then the exponent α may be approximated by

$$\alpha_N = d - 2 - \frac{A_N^{(d-1)}(\lambda_{c,N})}{DA_N^{(d)}(\lambda_{c,N})}\,.$$

It is worth noting that the formulae for the location and the exponent of the dominant singularity involve only the coefficients of the highest derivatives in the differential equation that defines the approximant. This motivates the choice (11), with its emphasis on those very coefficients.

Since d is not fixed as N increases, the polynomials $A_N^{(k)}$ that define the new method cannot be computed by using Sergeyev's recurrence relation; and so the system of equations for the polynomials will be solved by Gaussian elimination.

Once the polynomials have been found, we can, if need be, solve the differential equation for the approximant u_N numerically or by using the standard Frobenius method, if it converges.

3 Some applications

It is not clear how to obtain asymptotic error estimates for the new method. Instead, we shall apply it to some examples for which we can compute the error directly. In this way, we can at least gain some insight into the effectiveness of the method.

Example 1 Let $a \in \mathbb{C}$. The polylogarithmic function $P(a, \lambda)$ is defined by

$$P(a,\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n^a} \quad \text{for } |\lambda| < 1, \tag{13}$$

and by analytic continuation elsewhere. There is a branch point at $\lambda = 1$ and one usually takes the branch cut along the half line $(-\infty, 1)$. There is a countable infinity of branches [1].

By differentiating the series term by term, we obtain the trivial identity

$$\lambda\, DP(a,\lambda) = P(a-1,\lambda)\,. \tag{14}$$

Now,

$$P(0,\lambda) = \frac{\lambda}{1-\lambda}$$

and so

$$1 - (1-\lambda)DP(1,\lambda) = 0.$$

Hence, we conclude that, for $a = 1$, the high-order approximant obtained by taking $d = 2$ is *exact* for the series $U = P(1,\lambda)$. More generally, in view of Equation (14), it is easily seen by induction on a that, for $a \in \mathbb{N}$, the high-order approximant obtained by taking $d = a+1$ is exact for the series $U = P(a,\lambda)$.

In the next example, we compare the high-order method with others for a number of "artificial" series.

Example 2 We consider seven test functions [10] with different types of singularities:

(I) **Additive algebraic singularities with the same exponent:**

$$u(\lambda) = 2(1-\lambda/3)^{-1/2} + 3(1-\lambda/4)^{-1/2} + 4(1-\lambda/5)^{-1/2} + 5(1-\lambda/6)^{-1/2}\,.$$

(II) **Additive algebraic singularities with different exponents:**

$$u(\lambda) = 2(1-\lambda/3)^{-1/2} + 3(1-\lambda/4)^{-1/3} + 4(1-\lambda/5)^{-1/4} + 5(1-\lambda/6)^{-1/5}\,.$$

(III) **Confluent algebraic/logarithmic singularity:**

$$u(\lambda) = \exp(\lambda)(1-\lambda/3)^{-1/2}\ln(1-\lambda/3)\,.$$

(IV) **Algebraic dominant singularity with a secondary logarithmic singularity:**

$$u(\lambda) = \exp(\lambda)(1-\lambda/3)^{-1/2}\ln(1-\lambda/4)\,.$$

(V) **Essential singularity:**

$$u(\lambda) = \exp[2(1-\lambda/3)^{-1/2}]\,.$$

Table 1

The number ρ_N (15) of correct decimal figures in the approximation of λ_c by various methods for the functions of Example 2.

u	N	High-order	Drazin–Tourigny	Padé	Neville [8]	Hunter–Guerrieri [10]
(I)	20	Exact	1.55	2.19	3.18	1.72
(II)	27	8.47	1.79	2.43	2.63	2.47
(III)	20	Exact	1.59	2.22	3.25	2.89
(IV)	20	Exact	1.09	1.94	3.69	1.54
(V)	14	Exact	1.73	1.41	1.72	1.39
(VI)	44	Exact	1.32	2.58	4.49	2.77
(VII)	44	11.56	1.12	2.59	4.00	2.48

(VI) **Confluent algebraic singularity:**

$$u(\lambda) = \exp(\lambda)[(1-\lambda/3)^{-1/2}]\{1+\sin\lambda(1-\lambda/3)^{1/3}\}\,.$$

(VII) **Another case of additive algebraic singularities:**

$$u(\lambda) = \exp(\lambda)[(1-\lambda/3)^{-1/2}]\{1+\sin\lambda(1-\lambda/4)^{1/3}\}\,.$$

The results of approximating the dominant singularity in each case by various methods of series analysis are shown in Table 1, where we give the number ρ_N of correct decimal figures; this is defined by

$$\rho_N = -\log_{10}\left|\frac{\lambda_c-\lambda_{c,N}}{\lambda_c}\right|, \tag{15}$$

where λ_c is the exact location of the dominant singularity and $\lambda_{c,N}$ is the estimate obtained by using N series coefficients. Here, the value of N is rather small, and so one should be careful not to infer too much from the evidence. Nevertheless, it is interesting to note how badly the Drazin–Tourigny method compares with the others. In most of the cases, we see that by using only

Table 2

Estimates of λ_c and α obtained from homogeneous high-order differential approximants for this Ising model problem.

d	N	λ_c	α
5	20	0.4142638326	0.2457038
6	27	0.4142183294	0.2493996
7	35	0.4142133966	0.2500453
8	44	0.4142135502	0.2500051
9	54	0.4142135620	0.2500002
exact		0.4142135623...	0.2500000

a small number of series coefficients, the high-order differential approximant produces exact results. The same is true for the critical exponent.

Example 3 In this example, we consider the two dimensional Ising model on a quadratic lattice, with ferromagnetic nearest neighbour interactions [14]. It is known that, for this example, $\lambda_c = \sqrt{2} - 1$ and $\alpha = 1/4$. Nickel computed the first 55 terms of the series. With these, by using second-order differential approximants, Gartenhaus and McCullough [7] obtained the biased estimate $\alpha \approx 0.2485$ for the exponent. The results of using high-order differential approximants are shown in Table 2. The accuracy obtained by using all the known terms of the series is very satisfactory.

Example 4 Consider the inviscid Burgerś equation in one-dimensional space:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{16}$$

with a periodic boundary condition in $[0, 2\pi]$ and the initial condition

$$u(x,0) = u_0(x) := -A \sin x\,. \tag{17}$$

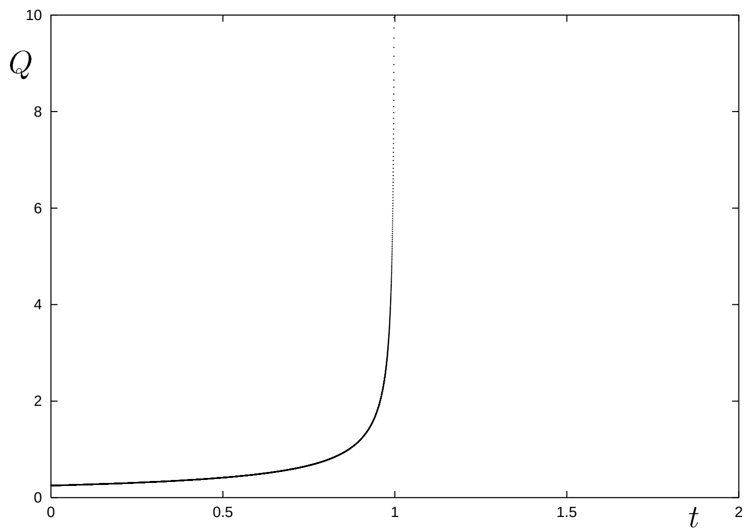

Fig. 1. The enstrophy Q against t for the solution of Burgerś equation with initial datum $u_0 = -\sin x$.

In this expression, A is a real parameter. Consider the series in powers of t for the so-called *enstrophy* Q defined by

$$Q(t, A) = \frac{1}{4\pi} \int_0^{2\pi} \left(\frac{\partial u}{\partial x} \right)^2 dx \,. \tag{18}$$

By direct calculation, we easily find the first few terms of the series:

$$Q(A, t) = A^2 \left[\frac{1}{4} + \frac{3}{16} A^2 t^2 + \frac{5}{32} (A^2 t^2)^2 + \frac{35}{256} (A^2 t^2)^3 + \ldots \right].$$

Set $\xi = t^2$ so that the series is now in powers of ξ. For $A = 1$, we find by using high-order differential approximants that Q satisfies the differential equation

$$4\xi(\xi - 1) \frac{dQ}{d\xi} + (6\xi - 4)Q = -1$$

with the initial condition

$$Q(0) = \frac{1}{4} \,.$$

Hence the dominating singularity of the enstrophy series for $A = 1$ is

$$Q(1, t) = \frac{1}{2t^2} \Big[(1 - t^2)^{-1/2} - 1 \Big] \quad \text{as } t \to 1.$$

In fact, algebraic approximants with $d \geq 2$ and N large enough are also exact for this problem since

$$\xi(1-\xi)Q^2 + (1-\xi)Q - \frac{1}{4} = 0\,. \tag{19}$$

A plot of the enstrophy against t is shown in Figure 1.

Very few nonlinear differential equations have solutions that can be obtained in terms of elementary functions. Even when an explicit formula for the solution is known, it is often so complicated that its qualitative properties are obscured. Thus, for most nonlinear differential equations, it is necessary to have reliable techniques to determine the approximate behaviour of the solutions. The next example is taken from Bender & Orszag [3]. In this case, we analyse the location and nature of the dominant singularity by using high-order differential approximants.

Example 5 We consider the equation [3]

$$Du - \frac{u}{1-\lambda u} = 0, \quad u(0) = 1. \tag{20}$$

This equation can in fact be solved exactly in terms of the Lambert W function. The dominant singularity is a square root branch point at $\lambda_c = 1/\mathrm{e}$.

In Table 3, the accuracy of the new method in determining the critical parameters λ_c and α as N increases is shown. The results suggest that the error decreases superexponentially

$$\sigma_N = -\frac{\ln|e_N|}{N\ln N} \quad \text{as} \quad N \to \infty, \tag{21}$$

where error $e_N = U(\lambda) - u_N$. By contrast, as shown in the last column, the Padé method appears to converge relatively slowly for this problem.

Table 3

Relative errors ($e'_N = |e_N/U|$) and rate of convergence (21) for the leading singularity point and the corresponding exponent for the solution of the first order nonlinear differential equation (20).

d	N	$e'_{\lambda_c,N}$	$\sigma_{\lambda_c,N}$	$e'_{\alpha,N}$	$\sigma_{\alpha,N}$	$e'_{\lambda_c,N}$ Padé
3	9	$0.4005\,10^{-03}$	0.5631	$0.150\,10^{-01}$	0.698	0.157
4	14	$0.5624\,10^{-08}$	0.5288	$0.616\,10^{-06}$	0.605	$0.496\,10^{-1}$
5	20	$0.2642\,10^{-13}$	0.5483	$0.688\,10^{-11}$	0.602	$0.270\,10^{-1}$
6	27	$0.2785\,10^{-22}$	0.5245	$0.200\,10^{-19}$	0.561	$0.143\,10^{-1}$
7	35	$0.5798\,10^{-32}$	0.5256	$0.503\,10^{-29}$	0.554	$0.841\,10^{-2}$
8	44	$0.4430\,10^{-43}$	0.5295	$0.630\,10^{-40}$	0.552	$0.527\,10^{-2}$
9	54	$0.1928\,10^{-56}$	0.5296	$0.427\,10^{-53}$	0.548	$0.347\,10^{-2}$
10	65	$0.3082\,10^{-71}$	0.5321	$0.101\,10^{-67}$	0.547	$0.238\,10^{-2}$

4 Application to symmetric Jeffery–Hamel flows

We consider here the well-known classical fluid dynamics problem named after Jeffery (1915) and Hamel (1916) for the steady two-dimensional flow of an incompressible viscous fluid from a line source or sink at the intersection between two rigid plane walls. Jeffery–Hamel solutions are particular similarity solutions of the Navier–Stokes equations and are found by solving an ordinary differential equation. Fraenkel [6] described all these solutions in terms of elliptic functions. Sobey and Drazin [15] studied their bifurcations theoretically and experimentally. They showed that the symmetric solution, which is stable for low Reynolds numbers, undergoes pitchfork and Hopf bifurcations as the Reynolds number increases. Though the hydrodynamic stability and bifurcation analysis of these flows has been investigated by many mathematicians,

Table 4

Estimates of K_c and α obtained by the high-order homogeneous differential approximant method for the Jeffery–Hamel problem ($\beta = 1/10$).

d	N	$K_{c,N}$	α_N
3	9	5.4580567148199421138	.57392089011191833535044
4	14	5.4580567148199421138	.50084769164770343420842
5	20	5.4581088584976777508	.49999442816686302680807
6	27	5.4581086860760345143	.50000001972260341647 48
7	35	5.4581086861111903811	.50000000000013613226997
8	44	5.4581086861111918638	.49999999999999999595014
9	54	5.4581086861111918638	.50000000000000000000000

the performance of seminumerical approximants remains to be found.

Let (r, θ) be polar coordinates, with $r = 0$ as the sink or source. Let β be the semi-angle and the domain of the flow be $-|\beta| < \theta < |\beta|$. Consider the velocity components u and v in the radial and tangential direction respectively.

We shall assume a symmetric radial flow, so that $v = 0$. Then the volumetric flow rate through the channel is

$$Q = \int_{-\beta}^{\beta} ur \, d\theta. \tag{22}$$

Let $\psi = \psi(r, \theta)$ be the stream function, then

$$\partial\psi/\partial\theta = ur, \quad \partial\psi/\partial r = 0.$$

A Reynolds number Re for the flow can be defined by $Re = Q/\nu$, where ν is kinematic viscosity. Expressed in terms of the dimensionless variables $y = \theta/\beta$ and $G(y; Re, \beta) = \psi(\theta)/Q$, the corresponding Navier–Stokes equations can be reduced to the ordinary differential equation [15]

$$G'''' + 4\beta^2 G'' + 2\beta Re\, G'G'' = 0, \tag{23}$$

with the boundary conditions

$$G = \pm 1, \qquad G' = 0 \qquad \text{at} \qquad y = \pm 1. \tag{24}$$

MAPLE can be used to find a sufficient number of coefficients of the series G in powers of Re and β. The first few coefficients are

$$G(\beta, Re) = \frac{1}{2}y(y^2-3) - \frac{3}{280}y(y^2-5)(y-1)^2(y+1)^2 Re\beta + \\ \frac{1}{10}y(y-1)^2(y+1)^2\beta^2 + \frac{1}{1400}y(5y^4-22y^2+33)(y-1)^2(y+1)^2 Re\beta^3 + \ldots. \tag{25}$$

To investigate this Jeffery–Hamel flow by approximant methods, we introduce the variable $g = G'$ and the parameter

$$K = \beta Re\,.$$

We use high-order differential approximants for the expansion of $g'(1)$ in powers of K with $\beta = 1/10$.

The dominating singularity of the series for $g'(1)$ at $\beta = 1/10$ appears to be of the form

$$g'(1) \sim C\,(K - K_c)^{\alpha} \quad \text{as } K \to K_c, \tag{26}$$

where C is a constant, K_c is the location of the singularity, and $\alpha \approx 1/2$. Some estimates $K_{c,N}$ and α_N obtained by the high-order differential approximant method are shown in Table 4. Other calculations suggest that the exponent α in equation (26) does not depend on the particular value of $\beta < 1$.

This is consistent with Fraenkel's asymptotic result

$$Re_c \sim \frac{5.461}{\beta} \quad \text{as} \quad \beta \to 0, \tag{27}$$

where our calculation shows that

$$Re_c \sim \frac{5.4581086861111918638}{\beta} \quad \text{as} \quad \beta \to 0.$$

5 Conclusion

In this paper, we have proposed a new form of differential approximant.

This *high-order differential approximant* is such that the order of the differential equation increases with N.

We have applied this method to a number of interesting series that have appeared in the literature. Somewhat to our surprise, we found that the method is *exact* in many cases. These include the series for

- the polylogarithmic function, which implies that the method is able to reveal the dominant behaviour of physical problems containing the branch point singularities related to polylogarithmic functions;
- test functions related to different types of singularities, which represent most of the series encountered in modern ratio-like methods for series analysis;
- the enstrophy of the solution of Burgerś equation. In fact Burgerś equation is a model problem to understand the blow-up behaviour in fluid motions.

We have also applied the new method to series where the form of the singularity is not known with certainty, such as the problem of Jeffery and Hamel [15], and we have compared the results with those obtained by other methods. Generally, we have found that the new method is very competitive. However, we have not yet developed a theory that would explain its strengths and limitations, and so we have relied on intelligent numerical investigations.

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References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, 1964) .

[2] G.A. Baker and P. Graves–Morris, *Padé Approximants* (Cambridge University Press, 1996) .

[3] C. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw–Hill, NewYork, 1978).

[4] P.G. Drazin and Y. Tourigny, Numerical study of bifurcation by analytic continuation of a function defined by a power series, *SIAM J. Appl. Math.* **56** (1996) 1-18.

[5] P.G. Drazin and Y. Tourigny, Extending the utility of perturbation series in problems of waves and stability, *Rend. Circ. Mat. Palermo, Series II, Suppl.* **45** (1996) L81-L84.

[6] L.E. Fraenkel, Laminar flow in symmetrical channels with slightly curved walls I: On the Jeffery–Hamel solutions for flow between plane walls *Proc. Roy. Soc. London A* **267** (1962) 119.

[7] S. Gartenhaus and W.S. McCullough, Higher order corrections for the quadratic Ising lattice susceptibility, *Phys. Rev. B* **38** (1988) 11689-11703.

[8] A.J. Guttmann, Asymptotic Analysis of Power Series Expansions, (Academic Press, 1989).

[9] A.J. Guttmann and G.S. Joyce, On a new method of series analysis in lattice statistics, *J. Phys. A: Gen. Phys.* **5** (1972) L81-L84.

[10] C. Hunter and B. Guerrieri, Deducing the properties of singularities of functions from their Taylor series coefficients, *SIAM J. Appl. Math.* **39** (1980) 248-263.

[11] D.L. Hunter and G.A. Baker, Methods of series analysis III: Integral approximant methods, *Phys. Rev. B* **19** (1979) 3808-3821.

[12] M.A.H. Khan, Singularity Analysis by Summing Power Series, *PhD thesis* (University of Bristol, 2001).

[13] M.A.H. Khan, Y. Tourigny and P.G. Drazin, A case study of methods of series summation: Kelvin-Helmholtz instability of finite amplitude, *To be published* (2001).

[14] M. Lévy, J.–C. Le Guillou and J. Zinn–Justin, editors, Phase Transitions: Cargése 1980, (NATO Advanced Study Institute, Series B, Plenum, New York, 1981).

[15] I.J. Sobey and P.G. Drazin, Bifurcation of two dimensional channel flows, *J. Fluid Mech.* **171** (1986) 263-287.

[16] A.V. Sergeyev, A recursive algorithm for Padé–Hermite approximations, *U.S.S.R. Comput. Math. Math. Phys.* **26** (1986) 17-22.

[17] A.V. Sergeyev and D.Z. Goodson, Summation of asymptotic expansions of multiple valued functions using algebraic approximants: Application to anharmonic oscillators, *J. Phys. A: Math. Gen.* **31** (1998) 4301-4317.

[18] Y. Tourigny and P.G. Drazin, The dynamics of Padé approximation, *Submitted to Nonlinearity* (2001).